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Flavor dependence of normalization constant for an infrared renormalon

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An ansatz is proposed for the flavor dependence of the normalization constant for the first IR renormalon in heavy quark pole mass.

The ultraviolet (UV) and infrared (IR) renormalons in field theory are fascinating objects that give rise to factorially growing large order behavior in perturbation theory. In quantum chromodynamics (QCD) the IR renormalons can be understood within the framework of operator product expansion (OPE) [1–6]. An IR renormalon in a Wilson coefficient in the OPE causes ambiguity in the resummed asymptotic series of the Wilson coefficient, which is to be cancelled by the vacuum condensate of an higher dimensional operator. This cancellation of the ambiguities in the resummation and vacuum condensate determines the nature of the renormalon singularities in the Borel plane. Specifically, the functional form of the condensate in the strong coupling α_s, which can be determined up to an overall constant by renormalization group (RG) equation, determines the power of singularity as well as its location [7]. However, the residue of the singularity, which is the normalization constant of the renormalon-caused large order behavior, is not known, but it can be calculated by a perturbation method [8, 9]. While the normalization can be expressed in a convergent series, with finite order perturbation it can only be calculated approximately. Its exact form is a nonperturbative quantity and so far there is no known way to calculate it. It would thus be very interesting if we somehow find the exact form for the normalization.

Our purpose in this Letter is to present an ansatz for the normalization for the first IR renormalon in the heavy quark pole mass. In this case the perturbation method for the normalization yields a series that converges rather quickly, which allows to determine the normalization within a few percent of uncertainty using the known first three perturbative coefficients [10, 11]. The reason for this rapid convergence and accuracy may lie with following two facts. First, the renormalon singularity is relatively soft, with its singularity of $(1-2b)^{-(1+\beta_1/2\beta_0^2)}$, for example, compared to that of the first IR renormalon in Adler function, which is of $(1-b/2)^{-(1+2\beta_1/\beta_0^2)}$, where β_0, β_1 are the first two coefficients of the beta function and b is the complex variable for the Borel plane. With vanishing flavor number $N_f = 0$, for example, the singularities are $(1-2b)^{-1.42}$ and $(1-b/2)^{-2.69}$, respectively. Secondly, the IR renormalon is the closest singularity to the origin in the Borel plane. This condition is important because the normalization in the perturbation method is evaluated on the boundary of convergence disk and for that to work the singularity should be closer to the origin than any other singularities. Of course, any renormalon singularity can be moved by a conformal mapping to be the closest one to the origin, this step is not required with the first IR renormalon for the pole mass, because it is already the closest one, and this seems to help the convergence. The normalization constant obtained from perturbation method was confirmed by recent lattice calculation of large order behaviour of static energy [12, 13].

To get the ansatz let us assume that the condensate in the OPE is of dimension n and is proportional to

$$\Lambda_{\rm QCD}^n \sim e^{-\frac{n}{2\beta_0\alpha_s}}\alpha^{-\frac{n\beta_1}{2\beta_0^2}}(1+\mathcal{O}(\alpha_s)) \tag{1}$$

and the associated renormalon singularity is of the form:

$$\frac{\mathcal{N}}{(1-b/b_0)^{1+\nu}}\,. \tag{2}$$

Then the imaginary part, which is ambiguous, of the Borel integral

$$\mathrm{Im}\left[\frac{1}{\beta_0}\int_0^\infty e^{-b/\beta_0\alpha_s}\frac{\mathcal{N}}{(1-b/b_0)^{1+\nu}}db\right] = \pm\mathcal{N}\sin(\nu\pi)\Gamma(-\nu)(b_0/\beta_0)^{1+\nu}e^{-b_0/\beta_0\alpha_s}\alpha_s^{-\nu}(1+\mathcal{O}(\alpha_s)) \tag{3}$$

is to be cancelled by ambiguity of the form (1). Thus identifying (3) with (1) we get

$$b_0 = n/2\,,\ \ \nu = n\beta_1/2\beta_0^2\,,\ \ \mathcal{N} = \frac{C(\frac{2\beta_0}{n})^\nu}{\sin(\nu\pi)\Gamma(-\nu)}\,, \tag{4}$$

where C is an unknown proportionality constant. Now putting

$$C = C_0 f(\nu) \tag{5}$$

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N_f	0	1	2	3	4	5
$\mathcal{N}_{\text{pert.}}$	0.6081	0.5981	0.5867	0.5730	0.5551	0.5288
$\mathcal{N}/\mathcal{N}_{\text{pert.}}$	1.0047	0.9979	0.9973	1.0081	1.0403	1.116

TABLE I. Comparison of the ansatz (9) with normalization from perturbation method for varying number of flavors.

with $f(0)=1$, where C_0 is determined so that the normalization agrees with that from the large-β_0 approximation at $\nu=0$, we get

$$\mathcal{N}=\frac{C_0 f(\nu)(\frac{2\beta_0}{n})^\nu}{\sin(\nu\pi)\Gamma(-\nu)}\,. \tag{6}$$

We shall now consider the normalization for the pole mass, for which $n=1$. While in this case the ambiguity in Borel summation is not canceled by a condensate of dimension one operator but by the static inter-quark potential [14], the ambiguity is nevertheless of the form (1) and the relations (4) are still valid. For this case the large-β_0 approximation fixes C_0 in the $\overline{\text{MS}}$ scheme as [15–17]

$$C_0=-C_F e^{5/6}\,, \tag{7}$$

where $C_F=4/3$ for QCD. Of course $f(\nu)$ is not known, and we can only guess it by comparing the normalization with values from the perturbation method. This gives

$$f(\nu)=4^{-\nu} \tag{8}$$

and, finally,

$$\mathcal{N}=-\frac{C_F e^{\frac{5}{6}}(\frac{\beta_0}{2})^\nu}{\sin(\nu\pi)\Gamma(-\nu)}\,, \tag{9}$$

for the $\overline{\text{MS}}$ scheme, where $\nu=\beta_1/2\beta_0^2$. This is our proposed ansatz.

How well does this ansatz work? Table I compares the ansatz with the normalization from the perturbative method for varying number of flavors, the details of which will be discussed shortly. The agreement is quite impressive for up to $N_f=4$. It is to be noted that the perturbative method tend to work better for smaller N_f. Of course, this does not mean that the ansatz is necessarily correct. One test may be to expand (9) for small ν and compare it with subleading corrections in large-β_0 approximation, for which only partial result exists [18]. Expanding (9) at $\nu=0$, we have

$$\mathcal{N}=\frac{C_F e^{5/6}}{\pi}[1+(\gamma_E+\log(\beta_0/2))\nu+\mathcal{O}(\nu^2)]\,. \tag{10}$$

The Euler constant term agrees with the corresponding term in [18]. It is worth noting that the two transcendental numbers π and γ_E in (10) both arise from

$$\sin(\nu\pi)\Gamma(-\nu) \tag{11}$$

in the denominator in (9). This is a strong indication that the term should be, at least, part of the exact form for the normalization.

It may be tempting to use the same idea to make an ansatz for other renormalons, for example, such as the first IR renormalon in the Adler function, which is associated with the gluon condensate, for which $n=4$. However, there is no accurate estimate of the normalization for the Adler function; The convergence from the perturbative method is not fast [19]. Without accurate numerical estimates for the normalization it is impossible to infer $f(\nu)$ for the Adler function and that blocks such an attempt. Nevertheless, if we assume (8) is applicable to the Adler function as well, then we have an ansatz for the first IR renormalon of the Adler function

$$\mathcal{N}=-\frac{\frac{3}{4}C_F e^{\frac{10}{3}}(\frac{\beta_0}{8})^\nu}{\sin(\nu\pi)\Gamma(-\nu)}\,, \tag{12}$$

where $\nu=2\beta_1/\beta_0^2$, and the following large-β_0 result is used [17, 20, 21]:

$$C_0=\frac{3}{4}C_F e^{\frac{10}{3}}\,. \tag{13}$$

It is interesting to see how the ansatz compares with perturbative results, even though the convergence for them is not that good and uncertainty is too large for accurate comparison. From [19], which uses five-loop Adler function, the normalization from perturbation is given as $0.287, 0.251, 0.208, 0.154$ for $N_f = 0, 1, 2, 3$, respectively, and the corresponding numbers from the ansatz are $0.327, 0.300, 0.282, 0.277$. As expected the ratios of corresponding numbers are not close to unity, but what is remarkable is that numbers from the ansatz and perturbation fall in the same ballpark. This is not so a trivial point because there is an order of magnitude difference between the large-β_0 result

$$\frac{e^{\frac{10}{3}}}{\pi} = 8.9 \tag{14}$$

and the numbers from the perturbative method, and there is no obvious reason that a function of such complex form as (12) should give values that are of same magnitude as the perturbative numbers.

We now show how $\mathcal{N}_{\text{pert.}}$ in Table I were obtained. The bilocal expansion, which interpolates the two expansions about the origin and the renormalon singularity, of the Borel transform $\tilde{m}(b)$ for the pole mass is given in the form [11, 22]:

$$\tilde{m}(b) = \sum_{n=0} \frac{h_n}{n!} \left(\frac{b}{\beta_0}\right)^n + \frac{\mathcal{N}}{(1-2b)^{1+\nu}}(1 + \sum_{i=1} c_i (1-2b)^i)\,, \tag{15}$$

with which the Borel summed pole mass m_{BR} is given by

$$m_{\text{BR}} = m_{\overline{\text{MS}}} \left[1 + \text{Re}\left(\frac{1}{\beta_0}\int_0^\infty e^{-b/\beta_0 \overline{\alpha}_s} \tilde{m}(b) db\right)\right]\,, \tag{16}$$

where $\overline{\alpha}_s \equiv \alpha_s(m_{\overline{\text{MS}}})$. The perturbative form of the Borel transform is

$$\tilde{m}(b) = \sum_{n=0} \frac{p_n}{n!} \left(\frac{b}{\beta_0}\right)^n\,, \tag{17}$$

which gives the perturbative expansion of the pole mass:

$$m_{\text{pole}} = m_{\overline{\text{MS}}}(1 + \sum_{n=0} p_n \overline{\alpha}_s^{n+1})\,, \tag{18}$$

where the first three coefficients are given as [23–25]

$$p_0 = 0.4244\,,\;\; p_1 = 1.3621 - 0.1055 N_f\,,\;\; p_2 = 6.1404 - 0.8597 N_f + 0.0211 N_f^2\,. \tag{19}$$

The normalization $\mathcal{N}$ is given by

$$\mathcal{N} = R(1/2)\,, \tag{20}$$

where

$$R(b) = \tilde{m}(b)(1-2b)^{1+\nu}\,, \tag{21}$$

and expanding $R(b)$ at the origin using (17) the normalization can be evaluated perturbatively [8, 9]. While this yields a convergent series for the normalization it does not fully exploit the bilocal expansion (15), especially the expansion about the renormalon singularity, of which the coefficients c_i are entirely dependent on the beta function [18], and with four-loop beta function the first two coefficients c_1, c_2 are known [10]. To utilize this expansion we write $R(b)$ in a truncated form:

$$R(b) = \left[\sum_{n=0}^{1} \frac{h_n}{n!} \left(\frac{b}{\beta_0}\right)^n\right](1-2b)^{1+\nu} + \mathcal{N}[1 + \sum_{i=1}^{2} c_i (1-2b)^i]\,, \tag{22}$$

and determine $h_0, h_1, \mathcal{N}$ by demanding (22) and (21) with (17) give identical expansion about the origin to $\mathcal{O}(b^2)$. To get the numbers in Table I this procedure was performed not in the $b-$plane but in the conformally mapped z-plane defined by

$$z = \frac{b}{1+b}\,, \tag{23}$$

where the main advantage of this mapping is that the first UV renormalon at $b = -1$ is mapped away to infinity. The first IR renormalon is now at $z = 1/3$ and the bilocal expansion of R in z-plane is given by

$$R(b(z)) = \left[\sum_{n=0}^{1} \frac{h_n^z}{n!}\left(\frac{z}{\beta_0}\right)^n\right](1-3z)^{1+\nu} + \mathcal{N}[1 + \sum_{i=1}^{2} c_i^z (1-3z)^i]\,, \tag{24}$$

from which $\mathcal{N}_{\rm pert.}$ in Table I were obtained. Note that c_1^z, c_2^z are given in terms of c_1, c_2 by

$$c_1^z = \frac{3}{2}c_1\,,\;\; c_2^z = \frac{9}{4}c_2 - \frac{3}{4}c_1\,. \tag{25}$$

In summary, we have proposed an ansatz for the normalization constant for the first IR renormalon in the pole mass and presented an argument that, at least, (11) should be an integral part of the normalization constant.

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